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RELATIVE INVARIANTS AND CLOSURE

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Part of the research program of The RAND Corporation consists of basic supporting studies in mathematics. This Memorandum presents a method for handling a problem that frequently arises in mathematical physics in connection with equations of heat, radiative transfer, etc., namely, that of replacing a system of nonlinear functional equations with one that is linear.

SUMMARY

One of the basic problems of mathematical physics is that of replacing a nonlinear functional equation by a more tractable (analytically and computationally) linear equation. More generally, one wants to replace a system of nonlinear equations, often derived from a single equation by means of an expansion in a parameter, an orthogonal expansion—with a system of linear functional equations. To treat this closure problem, the authors present a method based upon the concept of relative invariants and the use of the multidimensional Lagrange expansion theorem.

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RELATIVE INVARIANTS AND CLOSURE

1. INTRODUCTION

One of the basic problems of mathematical physics is that of replacing a nonlinear functional equation by an analytically and computationally more tractable linear equation. More generally, the problem is that of replacing a system of nonlinear functional equations by a system of linear functional equations. Often this system of nonlinear equations is derived from a single equation by means of an expansion in a parameter, or of an orthogonal expansion.

Consider, for example, the nonlinear heat equation

$$u_t = u_{xx} + u^3, \tag{1.1}$$

with the boundary and initial conditions

$$\begin{aligned} &u(x,0) = g(x), \quad 0 < x < 1, \\ &u(0,t) = u(1,t) = 0, \quad t > 0. \end{aligned} \tag{1.2}$$

If we set

$$u = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x, \tag{1.3}$$

substitution in (1.1) yields the infinite system of nonlinear equations

$$u_n'(t) = -n^2 u_n + \sum_{i,j=1}^{\infty} a_{ijn} u_i u_j + \cdots, \quad u_n(0) = g_n, \qquad n = 1,2,\ldots, \tag{1.4}$$

where

$$g(x) \sim \sum_{n=1}^{\infty} g_n \sin n\pi x.$$

We now face the problem of <u>closure</u> (see [1], [2]), which is to say of replacing (1.4) by a finite system of equations, preferably linear. We shall present a method based upon the concept of relative invariants and the use of the multidimensional Lagrange expansion theorem [2], [3]. In some cases, as in the case above, a preliminary use of approximation methods may be required.

2. FINITE-DIMENSIONAL CASE

In order to illustrate the idea of relative invariants, let us first consider a <u>finite</u> system of nonlinear differential equations, say

(2.1) $$\frac{dx_i}{dt} = -\lambda_i x_i + g_i(x), \quad x_i(0) = c_i, \quad i = 1,2,\dots,N,$$

where we suppose that

(2.2)(a) $\mathrm{Re}(\lambda_i) > 0$,

(b) $|c_i|$ is sufficiently small,

(c) $g_i(x)$ is a power series in the components of x beginning with quadratic terms.

We shall impose another condition upon the λ_i in a moment.

Our aim is to find N functions of the x_i, $\{\varphi_i(x)\}$, with the properties that

$$(2.3) \qquad \frac{d\varphi_i}{dt} = -\lambda_i \varphi_i,$$

and

$$(2.4) \qquad \varphi_i(x) = x_i + h_i(x),$$

where the $h_i(x)$ are power series in the components of x beginning with quadratic terms.

If such functions exist, we have, from (2.3),

$$(2.5) \qquad \varphi_i(x) = e^{-\lambda_i t} \varphi_i(c), \quad i = 1,2,\ldots,N.$$

Hence, the x_i are obtained as solutions of these N simultaneous equations. Since from (2.4) these equations have the form

$$(2.6) \qquad x_i = e^{-\lambda_i t} \varphi_i(c) - h_i(x), \quad i = 1,2,\ldots,N,$$

we are in an ideal position to employ the multidimensional Lagrange expansion to obtain a power series expansion for the x_i in terms of the quantities $e^{-\lambda_i t} \varphi_i(c)$. Cutting this off at any desired stage, we have the required closure.

Furthermore, if we so wish, we can employ the Lagrange expansion theorem to calculate N additional functions $\psi_i(x)$, $i = 1,2,\ldots,N$, in place of the N functions $x_1, x_2, \ldots, x_N$.

3. WHAT IS REQUIRED FOR THIS FORMALISM?

We mentioned above that a further condition upon the λ_i was required. In order to see how this arises, let us examine the problem of determining the functions $\varphi_i(x)$ of (2.3). Write

$$(3.1) \qquad \varphi_i(x) = x_i + a_{11i}x_1^2 + 2a_{12i}x_1x_2 + a_{22i}x_2^2 + \cdots,$$

and let us restrict ourselves to the case $i = 1$. We have

$$(3.2) \qquad \frac{d}{dt}\varphi_1(x) = \frac{dx_1}{dt} + 2a_{111}x_1\frac{dx_1}{dt} + 2a_{121}\frac{dx_1}{dt}x_2 + 2a_{121}x_1\frac{dx_2}{dt} + 2a_{221}x_2\frac{dx_2}{dt} + \cdots$$

$$= -\lambda_1x_1 + g_{111}x_1^2 + g_{121}x_1x_2 + \cdots$$
$$+ 2a_{111}x_1[-\lambda_1x_1 + g_{111}x_1^2 + g_{121}x_1x_2 + \cdots]$$
$$+ 2a_{121}x_2[-\lambda_1x_1 + g_{111}x_1^2 + g_{121}x_1x_2 + \cdots]$$
$$+ 2a_{121}x_1[-\lambda_2x_2 + g_{112}x_1^2 + \cdots]$$
$$+ 2a_{221}x_2[-\lambda_2x_2 + \cdots].$$

Let us concentrate solely upon first and second order terms, and, in particular, upon the determination of a_{111}, a_{121}, and a_{221}. We have, from (3.2),

$$(3.3) \qquad \frac{d}{dt}\varphi_1(x) = -\lambda_1x_1 + g_{111}x_1^2 + g_{121}x_1x_2 + g_{221}x_2^2 + \cdots - 2\lambda_1a_{111}x_1^2 - 2\lambda_1a_{121}x_1x_2 - 2\lambda_2a_{121}x_1x_2 - 2\lambda_2a_{221}x_2^2 + \cdots.$$

There are no other terms involving x_1^2, x_1x_2 or x_2^2. If (3.3) is to hold, for $i = 1$ we must have, upon equating coefficients,

$$(3.4)\qquad \begin{aligned} g_{111} - 2\lambda_1 a_{111} &= -\lambda_1 a_{111}, \\ g_{121} - (2\lambda_1 + 2\lambda_2)a_{121} &= -\lambda_1 a_{121}, \\ g_{221} - 2\lambda_2 a_{221} &= -\lambda_1 a_{221}. \end{aligned}$$

These relations determine the quantities a_{111}, a_{121}, a_{221} uniquely, provided that

$$(3.5)\qquad \lambda_1 \neq 0, \quad \lambda_1 + 2\lambda_2 \neq 0, \quad 2\lambda_2 - \lambda_1 \neq 0.$$

As far as the determination of all of the coefficients is concerned, we see that we have the condition

$$(3.6)\qquad m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k \neq 0,$$

for any nontrivial set of positive or negative integral or zero values of the m_i. This is a frequently met condition.

This condition is not merely a reflection of the weakness of the method. If, for example,

$$(3.7)\qquad \begin{aligned} x_1' &= -x_1, \quad x_1(0) = c_1, \\ x_2' &= -2x_2 + x_1^2, \quad x_2(0) = c_2, \end{aligned}$$

we see that a term of the te^{-2t} enters. The functions x_1 and x_2 are not power series in e^{-t} and e^{-2t}.

Observe that we face this situation in treating (1.4), since here $\lambda_n = n^2\lambda_1$.

4. PRELIMINARY CLOSURE

To avoid a violation of the condition in (3.6), we can use some preliminary closure techniques. Suppose, for example, that $2\lambda_2 - \lambda_1 = 0$, which means that the third equation in (3.4) is troublesome unless g_{221} and a_{221} are both zero. If $g_{221} = 0$, we may very simply set $a_{221} = 0$. If $g_{221} \neq 0$, we use an approximation technique to replace the $g_{221}x_2^2$ by a linear combination of other terms appearing in the expansion of $g_i(x)$; see [1].

Thus, we can set

$$(4.1) \qquad g_{221}x_2^2 \cong c_{11}x_1^2 + c_{12}x_2^2 + \cdots,$$

where the c_{ij} are to be determined by a least-squares fit. Various self-consistent methods can now be used.

5. INFINITE-DIMENSIONAL CASE

Consider now the infinite system of ordinary differential equations

$$(5.1) \qquad \frac{dx_i}{dt} = -\lambda_i x_i + g_i(x), \quad x_i(0) = c_i, \quad i = 1,2, \ldots,$$

obtained from a nonlinear operator equation

$$(5.2) \qquad u_t = Au + g(u),$$

where we have used an expansion for u which diagonalizes the operator A.

It is easy to see that if the condition of (3.6) is met, we can determine the coefficients of the functions $\varphi_i(x)$ recurrently, solving a single equation for a single unknown at each step. The determination of $\varphi_i(x)$ can be carried as far as desired. The stopping rule is now a closure technique. Having determined the N functions $\varphi_i(x)$, $i = 1,2,\ldots,N$, to some degree of accuracy, the N equations

$$(5.3) \qquad \varphi_i(x) = e^{-\lambda_i t}\varphi_i(c), \quad i = 1,2,\ldots,N,$$

may be solved as indicated above, or in some other way, to yield the x_i, $i = 1,2,\ldots,N$. We suppose in (5.3) that on the left-hand side, all terms involving x_i, $i > N + 1$, have been omitted.

There are many variants of this procedure, each equivalent to a different closure method.

6. APPLICATION TO THE NONLINEAR HEAT EQUATION

In applying this method to (1.4), we see that the first violation of (3.6) occurs in connection with the determination of fourth-order terms in the functions $\varphi_i(x)$ (since $\lambda_4 = 4\lambda_1$). Hence, if the deviation from linearity is small, or equivalently if $\max_x |g(x)|$ is small, the method sketched above will yield the second- and third-order correction terms for the solution of

$$u_1(x,t) = \sum_{n=1}^{\infty} g_n e^{-n^2 t} \sin n\pi x \tag{6.1}$$

in a convenient fashion.

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